

ON AN INEQUALITY RELATED TO A CERTAIN FOURIER COSINE SERIES

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ABSTRACT. We prove that the Fourier cosine series

$$\sum_{k=1}^{\infty} (-1)^{k+1} \frac{r^k \cos k\phi}{k+2}$$

assumes its maximum value at $\phi = 0$ for $\phi \in [0, \pi)$ regardless of r if $r \in (0, 1]$. This was first proved by Arias de Reyna and van de Lune. The more compact proof presented here is based on a generating function of the Chebyshev Polynomials.

1. REFORMULATION OF THE INEQUALITY

Arias de Reyna and van de Lune [2] established and proved the inequality

Theorem 1.1. *If $r \in (0, 1]$ and $\phi \in (0, \pi)$ then*

$$(1) \quad \sum_{k=1}^{\infty} (-1)^{k+1} \frac{r^k \cos k\phi}{k+2} < \sum_{k=1}^{\infty} (-1)^{k+1} \frac{r^k}{k+2}.$$

To get a simpler proof of this theorem it is convenient to eliminate the cosine function by setting $x := \cos \phi$. Then $-1 < x < 1$ and

$$(2) \quad \cos k\phi = T_k(\cos \phi) = T_k(x),$$

where $T_k(x)$ is the k -th Chebyshev Polynomial of the first kind. By this substitution we obtain the function of two variables

$$(3) \quad f(x, r) := \sum_{k=1}^{\infty} (-1)^{k+1} \frac{r^k T_k(x)}{k+2} \quad (r \in (0, 1], x \in (-1, 1])$$

from the left side of (1), where we have included the value $x = 1$ additionally. This function satisfies

Lemma 1.2. *$f(x, r)$ is a monotonically increasing function of x . So the inequality*

$$(4) \quad f(x, r) < f(1, r) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{r^k}{k+2} \stackrel{1}{=} \left(\log(1+r) - r + \frac{r^2}{2} \right) / r^2$$

holds for $r \in (0, 1]$ and $x \in (-1, 1)$.

From this, theorem 1.1 above follows immediately.

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¹ We have $T_k(1) = 1$ for all k .

2. AN INTEGRAL REPRESENTATION OF THE SERIES

As we can see in [2], trying to prove lemma 1.2 from definition (3) directly must be hard. So we are looking for a different representation of $f(x, r)$. Indeed, there is such a representation as a definite integral.

Lemma 2.1.

$$(5) \quad f(x, r) = \frac{1}{r^2} \int_0^r t^2 \frac{t+x}{t^2+2xt+1} dt.$$

Proof. The series in (3) looks similar to one of the generating functions of the Chebyshev Polynomials of the first kind. We take equation 22.9.9 from [1]²

$$\frac{1-xz}{1-2xz+z^2} = \sum_{k=0}^{\infty} T_k(x) z^k \quad (|z| < 1, |x| \leq 1).$$

After moving the constant term $T_0(x) \equiv 1$ of the series to the left side and setting $z = -r$ we obtain

$$r \frac{r+x}{r^2+2xr+1} = \sum_{k=1}^{\infty} (-1)^{k+1} T_k(x) r^k.$$

Multiplying this equation by r , writing t for r and integrating from 0 to r over t yields

$$\int_0^r t^2 \frac{t+x}{t^2+2xt+1} dt = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{r^{k+2}}{k+2} T_k(x) = r^2 f(x, r)$$

as required. Obviously this equation remains valid for $r = 1$ if $x \in (-1, 1]$. \square

So we get the

Proof of Lemma 1.2. Differentiation of (5) with respect to x shows that

$$\frac{\partial}{\partial x} f(x, r) = \frac{1}{r^2} \int_0^r t^2 \frac{1-t^2}{[t^2+2xt+1]^2} dt > 0$$

since the integrand is positive for $t \in (0, r)$. Therefore $f(x, r)$ is a monotonically increasing function of x and in particular inequality (4) must hold. The explicit expression of $f(1, r)$ follows from the logarithm series. \square

Thus theorem 1.1 is proved.

Corollary 2.2. *The integral in (5) can be solved, giving*

$$f(x, r) = \frac{1}{r^2} \left[\frac{r^2}{2} - xr + \left(x^2 - \frac{1}{2} \right) \log(r^2 + 2xr + 1) + 2xw \arctan\left(\frac{wr}{1+xr} \right) \right]$$

with the abbreviation $w := \sqrt{1-x^2}$.

But this representation does not help to prove lemma 1.2 since its derivative with respect to x takes on a very complicated shape.

² This can be shown by setting $z = r e^{i\phi}$ ($0 \leq r < 1$, $0 \leq \phi \leq \pi$) in the geometric series $1/(1-z) = \sum_{k=0}^{\infty} z^k$, taking the real part and using (2).

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